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Holographic chiral phase transition with chemical potential

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Abstract: We discuss the Sakai-Sugimoto model at finite temperature and finite chemical potential. It is a holographic model of large N_c QCD with N_f massless quarks based on a D4/D8-$\overline{\text{D8}}$ brane system. The near horizon limit of the D4-branes and the probe approximation of the D8-$\overline{\text{D8}}$ pairs allow us to treat the D4-branes as a gravitational background and the D8-$\overline{\text{D8}}$ pairs as a probe which does not affect the background. We propose that the asymptotic value of a U(1) gauge field on the D8-$\overline{\text{D8}}$-branes is identified with the chemical potential for the baryon number. Using this chemical potential we analyze the phase structure of this model and find a chiral symmetry phase transition of the first order.

Keywords: Gauge-gravity correspondence, Brane Dynamics in Gauge Theories, AdS-CFT Correspondence.

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Contents

1. Introduction

The AdS/CFT correspondence [1 – 3] (see [4] for a review) is a useful duality between a string theory in $(d+1)$-dimensional anti de Sitter spacetime (times a compact space) and a d-dimensional conformal field theory. The AdS/CFT correspondence can be extended to more general cases of the string/gauge duality for non-conformal and non-supersymmetric theories. In this scheme one can discuss some features of the low energy QCD such as confinement and spontaneous chiral symmetry breaking. Such an approach to low energy behaviors of QCD in terms of the string/gauge duality is often called the holographic QCD [5 – 10] (and references therein).

One of the interesting recent developments in the holographic approach to QCD is the D4/D8-$\overline{\text{D8}}$ model proposed by Sakai and Sugimoto [11, 12]. This brane system consists of N_c D4-branes compactified on S^1 and probe N_f D8-$\overline{\text{D8}}$-brane pairs. The D4-branes are described by the extremal brane solution in the near horizon limit with one of the spatial directions along its world-volume compactified on S^1. This gravitational background is dual to a five-dimensional gauge theory, which looks four-dimensional at energy scale below the compactification scale. This description is valid for the case $1 \ll g_{\text{YM}}^2 N_c \ll 1/g_{\text{YM}}^4$, where g_{YM} is the four-dimensional gauge coupling. Imposing periodic boundary conditions on the bosons and anti-periodic ones on the fermions along the compactified direction, supersymmetry is explicitly broken. The scalars and the fermions on the D4-branes become massive and are decoupled from the system at low energy. Thus one obtains a $\text{U}(N_c)$ pure gauge theory. To describe quarks in the fundamental representation of the gauge group $\text{U}(N_c)$ one introduces N_f D8-$\overline{\text{D8}}$ pairs into the D4 background. The probe approximation $N_f \ll N_c$ [5] allows us to treat the N_f D8-$\overline{\text{D8}}$ pairs as a probe, which does not affect the D4 background. A string connecting the D4-branes and the D8-branes ($\overline{\text{D8}}$-branes) represents a massless left-(right-)handed quark with N_f flavors. Therefore one obtains a four-dimensional $\text{U}(N_c)$ gauge theory with N_f flavored massless quarks in the fundamental

representation of the gauge group at low energy. The $U(N_f)_L \times U(N_f)_R$ symmetry of the D8 and $\overline{\text{D8}}$-branes represents a chiral symmetry of the quarks. It was shown in ref. [11] that this chiral symmetry is broken to a diagonal symmetry $U(N_f)_V$ because of a configuration of the probe D8-$\overline{\text{D8}}$ pairs. By the topology of the D4-brane geometry they must have a curved configuration in which the D8-branes and the $\overline{\text{D8}}$-branes are connected each other (see figure 2), which breaks the chiral symmetry.

The Sakai-Sugimoto model was also studied at finite temperature T [13 – 15]. There are two D4-brane geometries which represent a low temperature phase and a high temperature phase respectively. A phase transition between these phases occurs at a certain critical temperature T_c. This transition corresponds to a confinement/deconfinement transition in a dual gauge theory [16]. For each of the phases one can introduce probe D8-$\overline{\text{D8}}$ pairs. The only configuration of the D8-$\overline{\text{D8}}$ pairs which can realize in the low temperature phase $T < T_c$ is a curved D8-$\overline{\text{D8}}$ configuration as in the zero temperature case. Thus the chiral symmetry is always broken in the low temperature phase. On the other hand, another configuration exists in addition to the curved one in the high temperature phase $T > T_c$. The new configuration consists of straight D8-branes and $\overline{\text{D8}}$-branes disconnected each other (see figure 3). The chiral symmetry is unbroken for this configuration. A chiral symmetry phase transition can occur between these two configurations at a certain temperature [13, 14].

The purpose of the present paper is to analyze the Sakai-Sugimoto model at finite temperature T and finite baryon number chemical potential μ. We introduce a non-vanishing background U(1) gauge field on the probe D8-$\overline{\text{D8}}$-brane world-volume. The asymptotic value of this gauge field background is related to the baryon number chemical potential. There are several related works in which chemical potentials are introduced as asymptotic values of gauge fields [17 – 21]. The gauge fields and the chemical potentials considered there, however, are those for the R symmetry in the bulk geometry or for the isospin symmetry on the probe brane world-volume. In contrast, we consider the chemical potential for the baryon number $U(1)_V$ symmetry on the probe brane. The D8-$\overline{\text{D8}}$-brane configuration and the gauge field background are determined by equations derived from the Dirac-Born-Infeld effective action on the world-volume of the probe branes. By solving these equations and comparing values of the effective action for the solutions we discuss a chiral symmetry breaking as in refs. [11, 13, 14].

We can summarize our results as follows. In the low temperature phase $T < T_c$ of the confinement/deconfinement transition there is a unique solution for the probe brane configuration and the gauge field background. As in the case without the gauge field background the probe branes have a curved configuration and the chiral symmetry is always broken. In the high temperature phase $T > T_c$ there are two types of solutions. One solution has a curved brane configuration and the chiral symmetry is broken. The other solution has a straight brane configuration and the chiral symmetry is unbroken. A phase transition between these two solutions occurs at certain temperature and chemical potential. The transition is of the first order. There is a critical value of the chemical potential above which the phase transition never occurs for any temperature. A qualitative feature of the phase diagram is shown in figure 1.

The organization of this paper is as follows. In section 2 we review the bulk geometry

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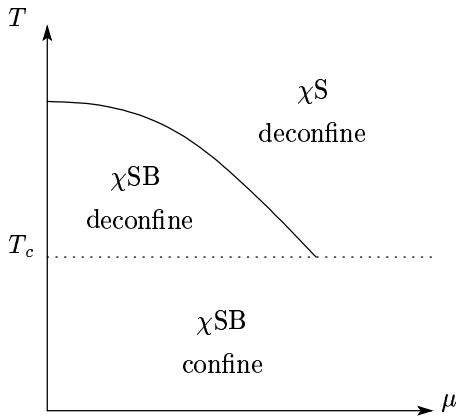

Figure 1: The phase diagram of the dual gauge theory.

of the Sakai-Sugimoto model at finite temperature and set up the effective action for the probe brane configuration and the U(1) gauge field. In sections 3 and 4 we obtain solutions for the brane configuration and the gauge filed and discuss the chiral symmetry breaking in the low and high temperature phases respectively. Section 5 is devoted to some discussions.

While preparing the manuscript of this paper, we have received a paper [22], in which a gauge field background on the probe branes is used to represent the baryon number chemical potential in studying hadronic matters in the Sakai-Sugimoto model. After submitting the manuscript of this paper for publication, we have received a paper [23] discussing the chiral phase transition in the D4/D8-$\overline{\text{D8}}$ model, in which an error in the first version of our paper was pointed out.

2. D4/D8-$\overline{\text{D8}}$ brane system

The Sakai-Sugimoto model [11, 12] is based on a D4/D8-$\overline{\text{D8}}$ brane system consisting of S^1 compactified N_c D4-branes and N_f D8-$\overline{\text{D8}}$-brane pairs transverse to the S^1. The brane configuration of the system is

	t	x^1	x^2	x^3	τ	U	θ^1	θ^2	θ^3	θ^4
D4	○	○	○	○	○	–	–	–	–	–
D8-$\overline{\text{D8}}$	○	○	○	○	–	○	○	○	○	○

(2.1)

with τ and θ's being coordinates of S^1 and S^4 respectively. The period of τ is denoted as $\delta\tau = 2\pi/M_{KK}$. In the large N_c limit and the near horizon limit the D4-branes are described by a bulk background geometry, which is a classical solution of the type IIA supergravity in ten dimensions. Assuming $N_f \ll N_c$ the D8-$\overline{\text{D8}}$ pairs are treated as a probe which does not affect the bulk background.

The finite temperature behavior of the Sakai-Sugimoto model was discussed in [13 – 15]. The bulk background geometry is represented by a metric with a periodic Euclidean time coordinate $t_E \equiv it \sim t_E + \delta t_E$ in addition to the periodic τ. The period of t_E is the inverse temperature $\delta t_E = 1/T$. There are two such Euclidean solutions which have an

appropriate asymptotic boundary behavior. One of them is the Euclidean version of the extremal D4-brane geometry compactified on S^1 with the metric

$$
\begin{aligned}
ds^2 &= \left(\frac{U}{R}\right)^{\frac{3}{2}} \left(dt_E^2 + \delta_{ij} dx^i dx^j + f(U) d\tau^2\right) + \left(\frac{R}{U}\right)^{\frac{3}{2}} \left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right), \\
f(U) &= 1 - \frac{U_{KK}^3}{U^3}, \qquad U_{KK} = \frac{4}{9} R^3 M_{KK}^2,
\end{aligned} \tag{2.2}
$$

where $d\Omega_4^2$ is the metric of S^4 and $R^3 = \pi g_s N_c l_s^3$ with g_s and l_s being the string coupling and the string length. The parameter U_{KK} must be related to M_{KK} as above to avoid a singularity of the metric at $U = U_{KK}$. With this relation the τ-U submanifold has a cigar-like form with a tip at $U = U_{KK}$. The dilaton ϕ and the RR 3-form C_3 are given by

$$
e^{\phi} = g_s \left(\frac{U}{R}\right)^{\frac{3}{4}}, \qquad F_4 = dC_3 = \frac{2\pi N_c}{V_4} \epsilon_4, \tag{2.3}
$$

where ϵ_4 and V_4 are the volume form and the volume of S^4. The other solution is the Euclidean version of the non-extremal D4-brane geometry

$$
\begin{aligned}
ds^2 &= \left(\frac{U}{R}\right)^{\frac{3}{2}} \left(\tilde{f}(U) dt_E^2 + \delta_{ij} dx^i dx^j + d\tau^2\right) + \left(\frac{R}{U}\right)^{\frac{3}{2}} \left(\frac{dU^2}{\tilde{f}(U)} + U^2 d\Omega_4^2\right), \\
\tilde{f}(U) &= 1 - \frac{U_T^3}{U^3}, \qquad U_T = \frac{16\pi^2}{9} R^3 T^2
\end{aligned} \tag{2.4}
$$

with the dilaton and the RR 3-form given in eq. (2.3). The parameter U_T must be related to T as above to avoid a singularity of the metric at $U = U_T$. The t_E-U submanifold has a cigar-like form with a tip at $U = U_T$. It is obvious that these two backgrounds are related by interchanging τ, U_{KK} and t_E, U_T.

It was shown [16, 8, 13] that the background (2.2) is dominant at low temperature, while (2.4) is dominant at high temperature by comparing values of the Euclidean supergravity action for these backgrounds. A phase transition between these backgrounds occurs at the temperature for $U_T = U_{KK}$, i.e. $T_c = M_{KK}/(2\pi)$. This phase transition is of the first order and represents a confinement/deconfinement transition [16].

In refs. [13, 14] N_f D8-$\overline{\text{D8}}$ pairs were introduced as a probe in the backgrounds (2.2), (2.4). The effective action of the D8-branes consists of the Dirac-Born-Infeld action and the Chern-Simons term

$$
S_{\text{D8}} = T_8 \int d^9x \, e^{-\phi} \, \text{Tr} \sqrt{\det(g_{MN} + 2\pi\alpha' F_{MN})} - \frac{i}{48\pi^3} \int C_3 \, \text{Tr} F^3, \tag{2.5}
$$

where g_{MN} and $F_{MN} = \partial_M A_N - \partial_N A_M - i\,[A_M, A_N]$ $(M, N = 0, 1, \cdots, 8)$ are the induced metric and the field strength of the $\text{U}(N_f)$ gauge field A_M on the D8-branes. T_8 is the tension of the D8-brane and $\alpha' = l_s^2$ is the Regge slope parameter. The effective action for the $\overline{\text{D8}}$-branes has a similar form. The total effective action has a gauge symmetry

$$
\text{U}(N_f)_L \times \text{U}(N_f)_R = \text{SU}(N_f)_L \times \text{SU}(N_f)_R \times \text{U}(1)_V \times \text{U}(1)_A, \tag{2.6}
$$

where $\mathrm{U}(N_f)_L$ and $\mathrm{U}(N_f)_R$ are symmetries of N_f D8 and $\overline{\mathrm{D8}}$-branes respectively. It was argued in ref. [11] that this gauge symmetry corresponds to a flavor chiral symmetry of quarks. The total effective action can be written in the form (2.5) with the integrations being over the whole of the D8-$\overline{\mathrm{D8}}$ world-volume. We use this form of the effective action in the following.

In refs. [11, 13] the gauge fields on the probe branes are treated as fluctuations representing the hadron spectrum. In this paper we consider a background gauge field. We assume that only the Euclidean time component of the U(1) gauge field has a non-vanishing background. We will see that it corresponds to an introduction of the baryon number chemical potential. We use a physical gauge for D8-brane world-volume reparametrizations and use the spacetime coordinates other than τ as the world-volume coordinates. Then, D8 and $\overline{\mathrm{D8}}$-brane configurations are determined by τ as a function of those world-volume coordinates. We make an ansatz that A_0 and τ depend only on the coordinate U

$$\tau = \tau(U), \qquad A_0 = A_0(U). \tag{2.7}$$

By this ansatz the Chern-Simons term in eq. (2.5) vanishes and does not concern us.

3. Low temperature phase

In the low temperature phase the geometry (2.2) is dominant. Using the ansatz (2.7) the induced metric g_{MN} on the probe D8-branes is given by

$$\begin{aligned} ds^2 = &\left(\frac{U}{R}\right)^{\frac{3}{2}} \left(dt_E^2 + \delta_{ij} dx^i dx^j\right) \\ &+ \left[\left(\frac{U}{R}\right)^{\frac{3}{2}} f(U)(\tau'(U))^2 + \left(\frac{R}{U}\right)^{\frac{3}{2}} \frac{1}{f(U)}\right] dU^2 + \left(\frac{R}{U}\right)^{\frac{3}{2}} U^2 d\Omega_4^2, \end{aligned} \tag{3.1}$$

where $\tau' = \frac{d\tau}{dU}$. Then, the effective action of the D8-branes (2.5) becomes

$$S_{\mathrm{D8}} = \frac{N_f T_8 V_4}{g_s} \int d^4x\, dU U^4 \left[f\,(\tau')^2 + \left(\frac{R}{U}\right)^3 \left(f^{-1} - \left(2\pi\alpha' A_0'\right)^2\right)\right]^{\frac{1}{2}}, \tag{3.2}$$

where $A_0' = \frac{dA_0}{dU}$.

This action leads to equations of motion for $\tau(U)$ and $A_0(U)$

$$\begin{aligned} &\frac{d}{dU}\left[\frac{U^4 f\,\tau'}{\sqrt{f\,(\tau')^2 + \left(\frac{R}{U}\right)^3 \left(f^{-1} - \left(2\pi\alpha' A_0'\right)^2\right)}}\right] = 0, \\ &\frac{d}{dU}\left[\frac{U^4 \left(\frac{R}{U}\right)^3 A_0'}{\sqrt{f\,(\tau')^2 + \left(\frac{R}{U}\right)^3 \left(f^{-1} - \left(2\pi\alpha' A_0'\right)^2\right)}}\right] = 0, \end{aligned} \tag{3.3}$$

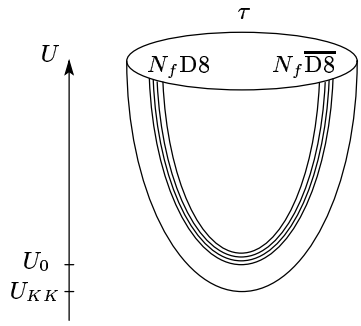

Figure 2: A D8-$\overline{\text{D8}}$-brane configuration in the low temperature phase.

which can be easily integrated once. We obtain

$$
\begin{aligned}
(\tau'(U))^2 &= \frac{\left(U_0^8 + C^2\left(\frac{U_0}{R}\right)^3\right) f(U_0)\left(\frac{R}{U}\right)^6}{f(U)^2\left[\left(\frac{R}{U}\right)^3\left(U^8 f(U) - U_0^8 f(U_0)\right) + C^2\left(f(U) - f(U_0)\left(\frac{U_0}{U}\right)^3\right)\right]}, \\
\left(2\pi\alpha' A_0'(U)\right)^2 &= \frac{C^2}{\left(\frac{R}{U}\right)^3\left(U^8 f(U) - U_0^8 f(U_0)\right) + C^2\left(f(U) - f(U_0)\left(\frac{U_0}{U}\right)^3\right)},
\end{aligned}
\tag{3.4}
$$

where C and U_0 are integration constants. As in the zero temperature case [11] and the low temperature phase in ref. [13] we have imposed a condition $\tau'(U_0) = \infty$. A typical configuration of $\tau(U)$ is shown in figure 2. Since there is no place for the D8 and $\overline{\text{D8}}$-branes to end, they are connected at $U = U_0$. We also impose the boundary condition $A_0(\infty) = \mu$ at the both ends of the D8-$\overline{\text{D8}}$ world-volume, where μ is a constant. We will identify this constant with the chemical potential for the baryon number later. Solving eq. (3.4) with this boundary condition we find for $U \sim U_0$

$$
A_0(U) \sim A_0(U_0) + \text{const.} \times C\,|U - U_0|^{\frac{1}{2}}. \tag{3.5}
$$

This solution is singular at $U = U_0$ and does not actually satisfy the original equation (3.3) unless $C = 0$. Therefore, we must choose $C = 0$ and obtain $A_0(U) = \mu$. Because of the connected configuration of the D8 and $\overline{\text{D8}}$-branes the chiral symmetry $\text{U}(N_f)_L \times \text{U}(N_f)_R$ on the probe D8-$\overline{\text{D8}}$ pairs is always broken to a diagonal subgroup $\text{U}(N_f)_V$ in the low temperature phase. The situation is the same as in the cases without the gauge field background [11, 13].

Instead of using the constant U_0 to parametrize the solution we can also use the $U = \infty$ asymptotic separation L between the D8 and $\overline{\text{D8}}$-branes in the τ-direction. It is related to U_0 by

$$
L = 2\int_{U_0}^{\infty} dU\,\tau'(U), \tag{3.6}
$$

where $\tau'(U)$ is given in eq. (3.4) with $C = 0$.

Substituting eq. (3.4) with $C = 0$ into the action (3.2) and introducing new variables $u = U/U_0$, $u_{KK} = U_{KK}/U_0$ and $f(u) = 1 - u_{KK}^3/u^3$ the effective action becomes

$$S_{\rm D8} = \tilde{T}_8 \int_1^\infty du\, u^5 \sqrt{\frac{u^3}{u^8 f(u) - f(1)}}, \tag{3.7}$$

where

$$\tilde{T}_8 = \frac{N_f T_8 V_4}{g_s} \left(R^3 U_0^7\right)^{\frac{1}{2}} \int d^4x. \tag{3.8}$$

Note that this reproduce the result in ref. [13].

4. High temperature phase

In the high temperature phase the geometry (2.4) is dominant. Using the ansatz (2.7) the induced metric on the probe D8-branes is

$$\begin{aligned} ds^2 = &\left(\frac{U}{R}\right)^{\frac{3}{2}} \left(\tilde{f}(U) dt_E^2 + \delta_{ij} dx^i dx^j\right) \\ &+ \left[\left(\frac{U}{R}\right)^{\frac{3}{2}} (\tau'(U))^2 + \left(\frac{R}{U}\right)^{\frac{3}{2}} \frac{1}{\tilde{f}(U)}\right] dU^2 + \left(\frac{R}{U}\right)^{\frac{3}{2}} U^2 d\Omega_4^2 \end{aligned} \tag{4.1}$$

and the effective action of the D8-branes (2.5) becomes

$$S_{\rm D8} = \frac{N_f T_8 V_4}{g_s} \int d^4x\, dU U^4 \left[\tilde{f}\,(\tau')^2 + \left(\frac{R}{U}\right)^3 \left(1 - \left(2\pi\alpha' A_0'\right)^2\right)\right]^{\frac{1}{2}}. \tag{4.2}$$

This action leads to equations of motion for $\tau(U)$ and $A_0(U)$

$$\begin{aligned} &\frac{d}{dU}\left[\frac{U^4 \tilde{f}\,\tau'}{\sqrt{\tilde{f}\,(\tau')^2 + \left(\frac{R}{U}\right)^3 \left(1 - (2\pi\alpha' A_0')^2\right)}}\right] = 0, \\ &\frac{d}{dU}\left[\frac{U^4 \left(\frac{R}{U}\right)^3 A_0'}{\sqrt{\tilde{f}\,(\tau')^2 + \left(\frac{R}{U}\right)^3 \left(1 - (2\pi\alpha' A_0')^2\right)}}\right] = 0, \end{aligned} \tag{4.3}$$

which can be easily integrated once as before. As in the case without the gauge field [13, 14] there are two types of solutions in the high temperature phase.

One solution is similar to the one in the low temperature phase. The integration of eq. (4.3) gives

$$\left(\tau'(U)\right)^2 = \frac{U_0^8 \tilde{f}(U_0)}{\left(\frac{U}{R}\right)^3 \tilde{f}(U) \left(U^8 \tilde{f}(U) - U_0^8 \tilde{f}(U_0)\right)}, \qquad A_0(U) = \mu. \tag{4.4}$$

where U_0 is an integration constant. As before we have imposed the boundary conditions $\tau'(U_0) = \infty$ and $A_0(\infty) = \mu$. A typical configuration of $\tau(U)$ is shown in figure 3 (a). The

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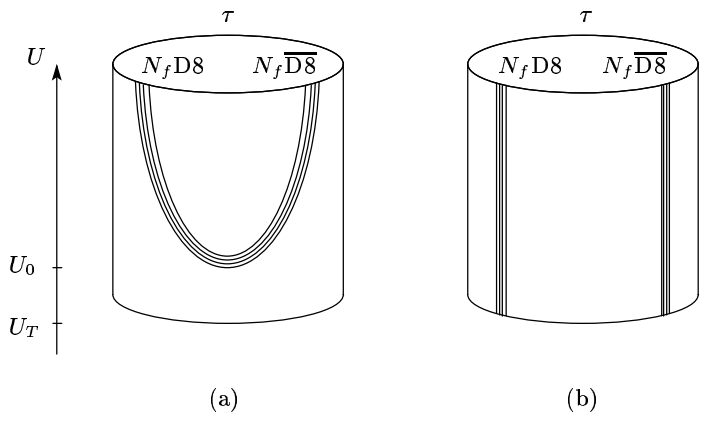

Figure 3: D8-$\overline{\text{D8}}$-brane configurations in the high temperature phase.

chiral symmetry $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ is broken to a diagonal subgroup $\mathrm{U}(N_f)_V$. Substituting eq. (4.4) into eq. (4.2) the effective action becomes

$$S^{\mathrm{U}}_{\mathrm{D8}} = \tilde{T}_8 \int_1^\infty du\, u^5 \sqrt{\frac{u^3 \tilde{f}(u)}{u^8 \tilde{f}(u) - \tilde{f}(1)}}, \tag{4.5}$$

where we have rescaled the variables as $u = U/U_0$, $u_T = U_T/U_0$, $\tilde{f}(u) = 1 - u_T^3/u^3$, and $\tilde{T}_8$ is given in eq. (3.8).

Instead of using U_0 we can also use the asymptotic separation L in eq. (3.6) to parametrize the solution, which is more convenient when comparing this solution to the other one. The relation between L and U_0 is obtained from eqs. (3.6) and (4.4) as

$$L = \left(\frac{R^3}{U_0}\right)^{\frac{1}{2}} F(u_T), \tag{4.6}$$

where

$$F(u_T) = 2 \int_1^\infty du \sqrt{\frac{\tilde{f}(1)}{u^3 \tilde{f}(u) \left(u^8 \tilde{f}(u) - \tilde{f}(1)\right)}}. \tag{4.7}$$

For the other solution the first integration of eq. (4.3) gives

$$\tau'(U) = 0, \qquad \left(2\pi\alpha' A_0'(U)\right)^2 = \frac{C^2}{U^8 \left(\frac{R}{U}\right)^3 + C^2}, \tag{4.8}$$

where C is an integration constant. $\tau'(U) = 0$ is the trivial solution of (4.3). A typical configuration is shown in figure 3 (b). It describes a situation that the probe D8 and $\overline{\text{D8}}$-branes separately extend along the U-direction in straight lines. The separation between the D8 and $\overline{\text{D8}}$-branes is chosen to be the same as the asymptotic separation L in the previous solution. The chiral symmetry $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ is unbroken in this case. Substituting eq. (4.8) into eq. (4.2) and using the rescaled variables as in eq. (4.5) the effective action becomes

$$S^{||}_{\mathrm{D8}} = \tilde{T}_8 \int_{u_T}^\infty du\, \frac{u^5}{\sqrt{u^5 + c^2}}, \tag{4.9}$$

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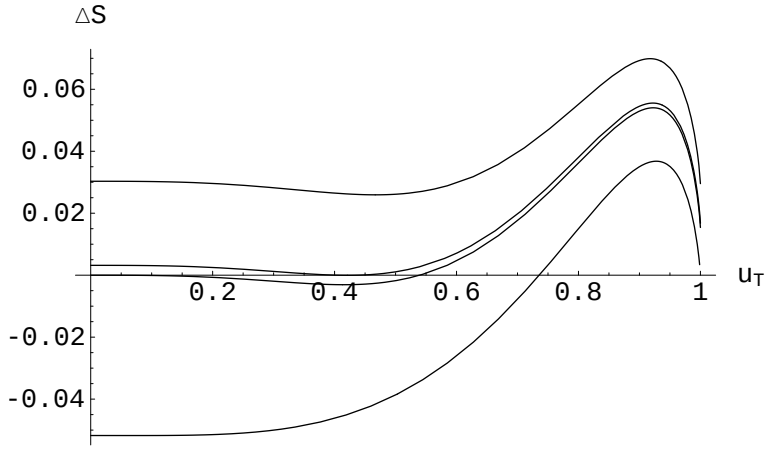

Figure 4: ΔS as a function of u_T for various values of c. From the bottom to the top each line represents the case for $c = 0$, 0.2158, 0.2252, 0.3 respectively.

where

$$c^2 = \frac{C^2}{R^3 U_0^5}. \tag{4.10}$$

To determine which of the two solutions is dominant we compare the values of the effective action. From eqs. (4.5), (4.9) we obtain the difference as

$$\begin{aligned}\Delta S &\equiv \frac{S_{\rm D8}^{\rm U} - S_{\rm D8}^{||}}{\tilde{T}_8} \\ &= \int_1^\infty du\, u^5 \left[\sqrt{\frac{u^3 \tilde{f}(u)}{u^8 \tilde{f}(u) - \tilde{f}(1)}} - \frac{1}{\sqrt{u^5 + c^2}}\right] - \int_{u_T}^1 du\, \frac{u^5}{\sqrt{u^5 + c^2}}.\end{aligned} \tag{4.11}$$

For $\Delta S < 0$ the curved configuration (4.4) is dominant and the chiral symmetry is broken, while for $\Delta S > 0$ the straight configuration (4.8) is dominant and the chiral symmetry is unbroken. Although the integrals in eqs. (4.5), (4.9) are divergent at $U = \infty$, the difference is finite due to the same asymptotic behaviors of $\tau(U)$ and $A_0(U)$. We evaluate eq. (4.11) by numerical calculations. For that purpose it is more convenient to change an integration variable to $z = u^{-3}$, which has a finite interval $0 \leq z \leq 1$ for $1 \leq u < \infty$. The result of the calculations is shown in figure 4. The behaviors of ΔS as a function of u_T for various values of c are given. The special case $c = 0$ reduces to the result in ref. [13]. In this case ΔS is positive for u_T larger than a certain value u_{T0} and negative for $u_T < u_{T0}$. The chiral symmetry is broken for $u_T < u_{T0}$ and unbroken for $u_T > u_{T0}$. The point $u_T = u_{T0}$ is a phase transition point. This phase transition is of the first order since two different configurations in figure 3 are possible at the transition point. As c increases, the transition point u_{T0} decreases. When $c > 0.2158$, there appears a new region near $u_T = 0$ in which $\Delta S > 0$. When $c > 0.2252$, ΔS is positive for all values of u_T and the chiral symmetry is always unbroken.

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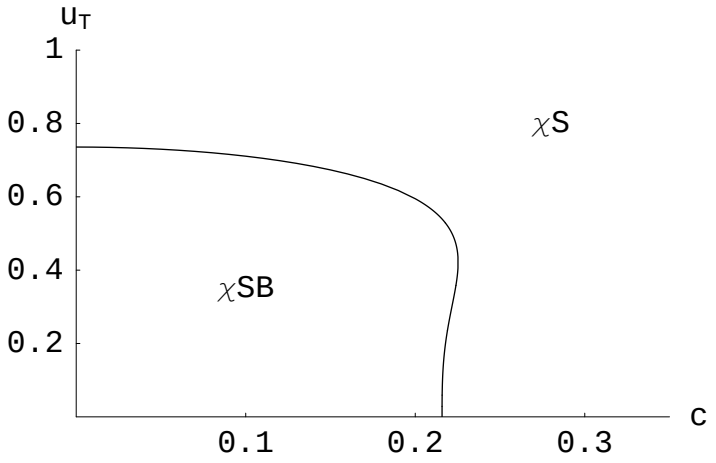

Figure 5: The phase diagram in the c-u_T space.

From these results we can draw a phase diagram in the c-u_T space as shown in figure 5. The chiral symmetry is broken in the region of small c and small u_T and unbroken outside of it. Note that we are considering here the high temperature phase of the confinement/deconfinement transition and only the part $u_T > u_{KK}$ of this diagram is valid.

It is more appropriate, however, to draw it in the space of the temperature T and the baryon number chemical potential μ. From eq. (2.4) the temperature T is related to u_T as

$$T = \frac{3}{4\pi}\left(\frac{U_0}{R^3}\right)^{\frac{1}{2}}\sqrt{u_T} = \frac{3}{4\pi}\frac{\sqrt{u_T}}{L}F(u_T), \tag{4.12}$$

where we have used eq. (4.6).

The relation of the chemical potential μ to u_T and c can be obtained as follows. From eq. (4.8) the large U behavior of $A_0(U)$ has a form

$$A_0(U) \sim \mu + \frac{v}{U^{\frac{3}{2}}}, \tag{4.13}$$

where μ and v are constants. We have chosen the same value μ for the constant term as in the curved solution (4.4). According to the AdS/CFT dictionary [4] for a massless vector field in a six-dimensional bulk, μ is a source coupled to an operator of dimension four $\mathcal{O}_4$ on a five-dimensional boundary. The U(1) gauge field A_0 defined on the whole of the D8-$\overline{\text{D8}}$ world-volume contains the gauge fields for both of $\text{U}(1)_V$ and $\text{U}(1)_A$ in the flavor symmetry (2.6). The part of A_0 which is symmetric for an interchange of D8 and $\overline{\text{D8}}$ corresponds to $\text{U}(1)_V$, while the part which is antisymmetric corresponds to $\text{U}(1)_A$ [11, 22]. Since the constant term μ is symmetric, it is a background value of the $\text{U}(1)_V$ gauge field coupled to the baryon number density $\mathcal{O}_4$, and μ is the baryon number chemical potential.

Integration of eq. (4.8) determines $A_0(U)$ up to a constant term (μ in eq. (4.13)). We can fix this constant term as follows. We first require that $A_0(U)$ vanishes at $U = U_T$

because of the regularity. To see this we first change the coordinates from (U, t_E) to (r, θ) defined by

$$U^3 = U_T^3 + U_T r^2, \qquad \theta = \frac{3}{2}\left(\frac{U_T}{R^3}\right)^{\frac{1}{2}} t_E. \tag{4.14}$$

From the induced metric (4.1) with $\tau'(U) = 0$ we see that (r, θ) are the polar coordinates near the point $U = U_T$. The point $U = U_T$ corresponds to the origin $r = 0$ and should be treated with care since the polar coordinates are not good coordinates near the origin. It is better to use the Cartesian coordinates

$$y = r\cos\theta, \qquad z = r\sin\theta. \tag{4.15}$$

The relation between A_0 and the components A_y, A_z in the coordinates (y, z) is obtained from $A_0 dt_E = A_y dy + A_z dz$ as

$$A_0 = \frac{3}{2}\left(\frac{U_T}{R^3}\right)^{\frac{1}{2}} r\left(-A_y \sin\theta + A_z \cos\theta\right). \tag{4.16}$$

Since we require that A_y and A_z are regular at the origin $r = 0$, $A_0(U)$ must vanish at $U = U_T$. We also note that although $A_0(U)$ is a gauge dependent quantity, it must vanish at $U = U_T$ in any gauge. Only the gauge transformations which preserve the condition $A_0(U_T) = 0$ are allowed.

The vanishing of $A_0(U)$ at $U = U_T$ fixes the constant term in this case and we find

$$A_0(U) = \frac{U_0}{2\pi\alpha'}\int_{u_T}^{u} du' \sqrt{\frac{c^2}{u'^5 + c^2}}. \tag{4.17}$$

The chemical potential μ is obtained as the asymptotic value for $U = \infty$

$$\mu = A_0(\infty) = \frac{R^3}{2\pi\alpha' L^2}\left(F(u_T)\right)^2 \int_{u_T}^{\infty} du \sqrt{\frac{c^2}{u^5 + c^2}}, \tag{4.18}$$

where we have used eq. (4.6) to eliminate U_0. This gives an expression of the chemical potential in terms of u_T and c.

Using eqs. (4.12), (4.18) we can convert the phase diagram in figure 5 to that in the μ-T space by numerical calculations. Using dimensionless variables

$$\tilde{T} = LT, \qquad \tilde{\mu} = \frac{2\pi\alpha' L^2}{R^3}\mu. \tag{4.19}$$

the phase diagram in the $\tilde{\mu}$-$\tilde{T}$ space is shown in figure 6. Only the part of this diagram for the high temperature phase of the confinement/deconfinement transition, i.e. $\tilde{T} > \tilde{T}_c = LM_{KK}/(2\pi)$ is valid. Therefore, our result of the phase diagram looks like figure 1 as we explained in Introduction. The orders of T and μ at the transition points can be estimated from eq. (4.19). Using $R^3 = g_{\rm YM}^2 N_c l_s^2/(2M_{KK})$ and $\tilde{\mu}, \tilde{T} = \mathcal{O}(1)$ we obtain

$$T = \mathcal{O}(L^{-1}), \qquad \mu = \mathcal{O}(g_{\rm YM}^2 N_c L^{-2} M_{KK}^{-1}). \tag{4.20}$$

If we assume $L = \mathcal{O}(M_{KK}^{-1})$, the transition temperature is of the order of the compactification scale M_{KK}, and the chemical potential is of the order $g_{\rm YM}^2 N_c M_{KK}$, which is much larger than M_{KK} since $g_{\rm YM}^2 N_c \gg 1$.

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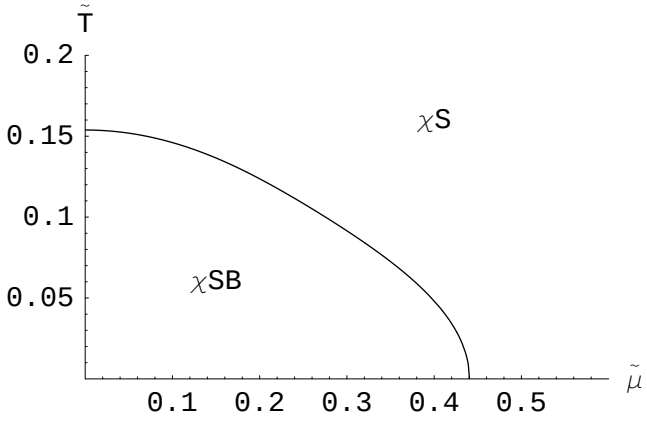

Figure 6: The phase diagram in the $\tilde{\mu}$-$\tilde{T}$ space.

5. Discussions

We analyzed the Sakai-Sugimoto model at finite temperature and finite baryon number chemical potential. The chemical potential is introduced as an asymptotic value of the U(1) gauge field on the probe D8-$\overline{\text{D8}}$-branes. Using this model we studied the phase structure of the chiral symmetry breaking and obtained the phase diagram in figure 1. This phase diagram should be compared with that expected in QCD [24]. Our result is especially different from the QCD expectation at low temperature. In QCD the chiral symmetry is expected to be unbroken even at zero temperature if the chemical potential is sufficiently large. In our analysis the chiral symmetry is always broken below T_c since the geometry of the τ-U space allows only the curved configuration of the probe branes. Since we used the probe approximation for the D8-$\overline{\text{D8}}$-branes, the gauge field on the branes (the chemical potential) does not affect the bulk geometry. It is interesting to see whether back-reactions of the gauge field on the geometry of the τ-U space change the phase structure below the temperature T_c. To fully understand the phase diagram we need an analysis beyond the probe approximation.

In the usual field theoretical approaches to the chiral symmetry breaking one considers condensations of quark bilinears $\bar{\psi}\psi$ as an order parameter. The quark masses are sources of these operators. In fact, in other models of the holographic QCD [7, 8] the mechanism of the chiral symmetry breaking is different from that in the Sakai-Sugimoto model. The chiral symmetry is realized as the rotational symmetry of the probe branes in the transverse space. The quark masses and the quark condensations can be read from asymptotic behaviors of the probe branes. In the Sakai-Sugimoto model quarks are always massless since the asymptotic distance between the D4-branes and the D8-$\overline{\text{D8}}$-branes, which is proportional to the quark mass, is zero. It is not clear how to introduce quark masses in this model. It is interesting to clarify the relation between the mechanisms of the chiral symmetry breaking in the Sakai-Sugimoto model and in other models.

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